Fun Problems
Geometry Research Honors

PROBLEMS

1. In quadrilateral $ABCD$, $\overline{AB} \perp \overline{BC}$, diagonal $\overline{AC} \perp \overline{CD}$, $AB = 9, BC = 3\sqrt{7}$, and $CD = 5$. Compute AD.

2. Line segments $\overline{PQ}, \overline{PR}$, and $\overline{PS}$ are three edges of a cube. If $PQ = 2$ and the area of ΔQRS is expressed in simplest form as $p\sqrt{q}$, compute $p + q$.

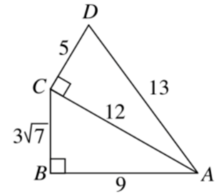
3. The coordinates of the vertices of quadrilateral ABCD are A(-3, 6), B(5, 8), C(1, -6), and D(-7, -4). If the consecutive midpoints of the sides of quadrilateral ABCD are joined to form a new quadrilateral, find the area of the new quadrilateral.

4. Quadrilateral WXYZ is a square whose side has a length of three. If square ABCD is inscribed in triangle XYZ with AD on XZ, B on XY, and C on ZY, find the area of square ABCD.

5. In equilateral triangle ABC with $AB = 8$, points P and Q are chosen on side $\overline{AB}$ so that $AP = BQ = 2$. Similarly, points R and S are chosen on side $\overline{BC}$ so that $BR = CS = 2$, and points T and U are chosen on side $\overline{CA}$ so that $CT = AU = 2$. If the area of hexagon $PQRSTU = H$, find H^2.

SOLUTIONS

1\.



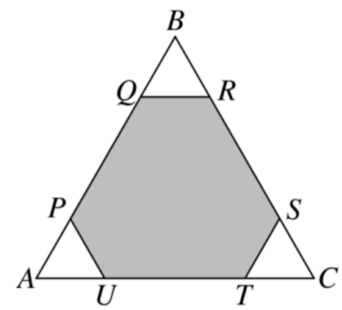
Use the Pythagorean Theorem in ΔABC to conclude that $AC = 12$, then use it again in ΔACD to find that $AD = 13$.

2\. Since each of $\overline{PQ}, \overline{PR}$, and $\overline{PS}$ is a diagonal of a face of the cube, ΔQRS is equilateral. In isosceles right ΔPQR, $QR = 2\sqrt{2}$, so the area of ΔQRS is $\frac{s^2\sqrt{3}}{4} = \frac{(2\sqrt{2})^2\sqrt{3}}{4} = 2\sqrt{3}$ and the required sum is **5**.

3\. 2. **48**. The coordinates of the midpoints are $(1,7), (3,1), (-3,-5)$, and $(-5,1)$. When consecutive midpoints of any quadrilateral are joined, the resulting quadrilateral is always a parallelogram. Notice that the diagonal of this parallelogram joining (-5,1) and (3,1) is a horizontal line. The area of the two congruent triangles above and below th diagonal can be calculated easily: 2[(1/2)(8)(6)] = 48 .

4\. **2.** Let $AB = x$. Then, $AB = XA = DZ = x$ and $ZX = 3x = 3\sqrt{2}$ So, the area of square $ABCD$=$(\sqrt{2})^2 = 2$.

5\.



Triangles APU, BQR, and CST are equilateral, and the area of each of them is 1/16 that of ΔABC. Thus, H is 13/16 the area of ΔABC. Use the formula $K = (s^2\sqrt{3})/4$ to find that the area of ΔABC is $16\sqrt{3}$. Then $H = 13\sqrt{3}$, so $H^2 = 507$.

Alternate solution: Draw $\overline{RU}$. Using $30° - 60° - 90°$ triangles, we can calculate the height of trapezoid $URQP$ to be $\sqrt{3}$ and the height of trapezoid $TSRU$ to be $2\sqrt{3}$. The required area is the sum of the areas of the trapezoids: $\frac{1}{2} \cdot (4+6) \cdot \sqrt{3} + \frac{1}{2} \cdot (2+6) \cdot 2\sqrt{3} = 13\sqrt{3}$.